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This thesis examines the structure of injective modules over commutative noetherian rings. The author shows that any injective module over a commutative noetherian ring can be written as a direct sum of indecomposable injective modules, where each summand is the injective hull of R/P for some prime ideal P. The results are then applied to the ring of integers.

INJECTIVE MODULES OVER COMMUTATIVE

NOETHERIAN RINGS

by

James Brooks Staton III

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Approved by

K. A. Byrd

Thesis Adviser

APPROVAL SHEET

This thesis has been approved by the following committee of the Faculty of the Graduate School at The University of North Carolina at Greensboro.

Thesis Advisor K. A. Byrd

Oral Examination Committee Members Karl Ray Gentry

E. E. Posey

Jerry E. Vaughan

29 July 1974

Date of Examination

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CHAPTER I

INTRODUCTION TO MODULES OVER A RING

This first chapter consists mainly of statements and definitions, and is intended to acquaint the uninitiated reader with a few facts about modules over a ring.

Definition: If R is an associative ring with identity 1, then an abelian group M is called a left R-module, provided there exits a mapping $\lambda: R \times M \to M$ such that the following conditions hold (for convience, we write ra in place of $\lambda(r,a)$):

(1) $r(a + b) = ra + rb$

(2) $rs(a) = r(sa)$

(3) $(r + s)(a) = ra + sa$

(4) $1a = a$

for $r, s \in R$ and $a, b \in M$. A right R-module is similarly defined. (If R is commutative, then M as a right R-module is equal to M as a left R-module, and will be referred to simply as an R-module, denoted M_k.)

Examples:

1. Any abelian group can be made into an R-module over the ring of integers by defining R-multiplication to be ra = the sum of r a's.
2. Given any ring R, an ideal of R can be made into an R-module by defining R-multiplication to be the usual multiplication of the ring. A special case of this example is

when R is an R-module over itself.

3. Any vector space is an R-module. In fact, the concept of a module is just a generalization of the concept of a vector space, except that the requirement for a field is reduced to a ring.

The requirements for a submodule of an R-module A are similar to the requirements for a subspace of a vector space.

Definition: Let A be an R-module, and let B be a non-empty subset of A. Then B is a submodule of A provided B is a subgroup of A and if $b \in B$, $r \in R$, then $br \in B$.

Thus if R is the ring of integers, and A is an abelian group, any subgroup of A is a submodule of A. But if R is the ring of rational numbers, and we consider R as a module over itself, then the subgroup of integers is not a submodule, since it is not closed under R-multiplication.

We can also define homomorphic mappings between two R-modules in a natural way.

Definition: Let A, B be two R-modules. By an R-module homomorphism we shall mean a mapping $\phi : A_R \to B_R$ such that ϕ is a group homomorphism of A into B and ϕ also satisfies the condition $\phi(ar) = \phi(a)r$ for all $a \in A$, $r \in R$.

The set of all such homomorphisms is usually denoted by $\mathrm{Hom}_R\,(A,B)$. This may be given the structure of an abelian group by defining addition as the natural addition of functions.

As with groups, we may also talk about factor modules, and the isomorphism theorems for groups can be easily restated for modules.

CHAPTER II

INJECTIVITY

Definition: A module M is called injective provided it has the following property: Let κ be monomorphism of some module A into some module B, then any homomorphism $\phi : A \to M$ can be "extended" to a homomorphism $\Psi : B \to M$ such that $\Psi \circ \kappa = \phi$.

$$\begin{tikzcd} M & B \arrow[l, dashed, "\psi"'] \\ & A \arrow[u, "\kappa"'] \arrow[ul, "\phi"] \end{tikzcd}$$

Theorem 2.1: If M is the direct product of a family of modules $\{M_i \mid i \in I\}$, then M is injective if and only if each M_i is injective.

Proof: ($\to$) We first note that if $M = \prod_{i \in I} M_i$, then there exists canonical epimorphisms $\Pi_i : M \to M_i$ and monomorphisms $\kappa_i : M_i \to M$ defined by

$$\Pi_i(m) = m(i), \quad \kappa_i(m_i)(j) = \begin{cases} m_i & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

(Lambek, pg. 18)

Clearly then

$$\Pi_j \circ \kappa_i = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

Now assume that each of the M_i are injective. Let $\kappa : A \to B$ be a monomorphism, and let $\Psi : A \to M$ be a given homomorphism. Since each M_i is injective, let ϕ_i be the extension of $\Pi_i \circ \Psi$ to B so that $\phi_i \circ \kappa = \Pi_i \circ \Psi$ for each $i \in I$.

$$\begin{tikzcd}
M_i & & B \arrow[ll, "\phi_i"'] \arrow[dl, dashed, "\phi"'] \\
& M \arrow[ul, "\Pi_i"] & \\
& & A \arrow[ul, "\psi"] \arrow[uu, "\kappa"']
\end{tikzcd}$$

Since M is a direct product of the M_i, there exists $\phi : B \to M$ such that ϕ is a homomorphism and $\Pi_i \circ \phi = \phi_i$ for each $i \in I$. Now $\phi_i \circ \kappa = \Pi_i \circ \Psi$ and $\Pi_i \circ \phi \circ \kappa = \phi_i \circ \kappa$ imply $\Pi_i \circ \phi \circ \kappa = \Pi_i \circ \phi$ for each $i \in I$. Thus $\phi \circ \kappa = \Psi$, and M is injective.

($\leftarrow$) Assume that M is injective, and choose any arbitrary M_i. Then if $\kappa : A \to B$ is a monomorphism, and if τ is a given homomorphism from A to M_i, then $\kappa_i \circ \tau : A \to M$ has an extension $\phi : B \to M$ so that $\phi \circ \kappa = \kappa_i \circ \tau$. If $\phi_i = \Pi_i \circ \phi$, then

$$\begin{aligned}
\phi_i \circ \kappa = \Pi_i \circ \phi \circ \kappa &= \Pi_i \circ (\phi \circ \kappa) \\
&= \Pi_i \circ (\kappa_i \circ \tau) \\
&= 1_{M_i} \circ \tau \\
&= \tau
\end{aligned}$$

And thus M_i is injective.

$$\begin{tikzcd}
M \arrow[dr, shift right, "\Pi_i"'] & & B \arrow[ll, "\phi"'] \arrow[dl, dashed, "\phi_i"] \\
& M_i \arrow[ul, shift right, "\kappa_i"'] & \\
& & A \arrow[ul, "\tau"] \arrow[uu, "\kappa"']
\end{tikzcd}$$

The following two theorems help characterize injectivity. The first theorem shows the relationship between an injective module M and the ideals of R. It makes the module R_R a decisive test module for injectivity.

Theorem 2.2 (Baer's Lemma): The R-module M is injective if and only if, for every right ideal K of R and every $\phi \in \mathrm{Hom}_R(K,M)$, there exists $m \in M$ such that $\phi(k) = mk$ for all $k \in K$.

Proof: See Lambek, pg. 88.

The next theorem shows the equivalence of injectivity with divisibility of abelian groups.

Theorem 2.3: An abelian group is injective if and only if it is divisible.

Proof: See Lambek, pg. 89.

Definition: A monomorphism $\kappa: M \to B$ is split provided there exists a homomorphism $\Pi: B \to M$ such that $\Pi \circ \kappa = 1$.

Theorem 2.4: The monomorphism $\kappa: M \to B$ is split provided the image $\kappa(M)$ of κ is a direct summand of B.

Proof: $(\to)$ Let $\kappa: M \to B$ be a monomorphism and assume $\kappa: M \to B$ is split. We show $B = \kappa(M) + \mathrm{Ker}\,\Pi$. Clearly $\kappa(M) \cap \mathrm{Ker}\,\Pi = 0$. Thus let $b \in B$, and show that $b \in \kappa(M) + \mathrm{Ker}\,\Pi$. Since $b \in B$, then $\Pi(b) = m$ for some $m \in M$. If $\Pi(b) = 0$, then $b \in \mathrm{Ker}\,\Pi$. Thus assume $m \neq 0$. Then $\kappa(m) = b' \in B$, and since $\Pi(\kappa(m)) = \Pi(b') = m$, $\Pi(b) = \Pi(b')$, or $\Pi(b - b') = 0$. Hence $b' \in \kappa(M)$, and $b - b' \in \mathrm{Ker}\,\Pi$. Since $b = b' + b - b'$, then $b \in \kappa(M) + \mathrm{Ker}\,\Pi$, and thus $\kappa(M)$ is a direct summand of B.

($\leftarrow$) Let $\kappa: M \to B$ be a monomorphism with $\kappa(M)$ a direct summand of B. Let $\Pi_1: B \to \kappa(M)$ be the projective epimorphism, and let $\kappa^{-1}: \kappa(M) \to M$ be the canonical isomorphism. Define $\Pi: B \to M$ as follows: $\Pi = \kappa^{-1} \circ \Pi_1$. Clearly Π is a homomorphism, and $\Pi \circ \kappa(m) = \kappa^{-1} \circ \Pi_1 \circ \kappa(m) = \kappa^{-1} \circ \kappa(m) = m$, or $\Pi \circ \kappa = 1$.

The next theorem states that a module is injective if and only if it is a direct summand of every module for which it is a submodule. In proving this, it is necessary to use the fact that every module is isomorphic to a submodule of an injective module.

Theorem 2.5: M is injective if and only if every monomorphism $\kappa: M \to B$ is split.

Proof: ($\to$) Assume M is injective, and let $\kappa: M \to B$ be a monomorphism. Then since M is injective, there exists $\Psi: B \to M$ such that $1 = \Psi \circ \kappa$. Hence κ is split.

($\leftarrow$) Assume every monomorphism $\kappa: M \to B$ is split. Then since every module is isomorphic to a submodule of an injective module, we may assume B is injective. Hence by our previous theorem, $\kappa(M)$ is a direct summand of B, and since B is injective, $\kappa(M)$ is injective, and thus M is injective.

Large submodules

Definition: Let M be an R-module. Then a submodule A of M is large provided it has a non-zero intersection with every non-zero submodule of M. We denote this by $A \subseteq' M$.

The following results follow immediately from the definition.

(1) If A_1, A_2 are large in B, then $A_1 \cap A_2 \subseteq' B$.

(2) If $A \subseteq' B$, and $B \subseteq' C$, then $A \subseteq' C$.

(3) If $A \subseteq B \subseteq C$ and $A \subseteq' C$, then $A \subseteq' B$.

(4) If $A \subseteq' B$ and $\phi \in \mathrm{Hom}_R(\kappa, B)$, then $\phi^{-1}(A) \subseteq' \kappa$.

In addition, we also have the following two theorems about large submodules.

Theorem 2.6: $M \subseteq' N$ if and only if given $x \in N \setminus 0$, there exists $r \in R$ such that $xr \neq 0$ and $xr \in M$.

Proof: $(\rightarrow)$ Assume $M \subseteq' N$, and let $x \in N \setminus 0$. Then xR is a submodule of N, $xR = 0$, so $M \cap xR \neq 0$. Hence there exists $a \in xR$ such that $a \in M$. But if $a \in xR$, then $a = xr$ for some $r \in R$. Therefore $xr \in M$.

$(\leftarrow)$ Let $M \subseteq N$, and assume for any $x \in N \setminus 0$, there exists $r \in R$ such that $xr \neq 0$ and $xr \in M$. Then for any submodule A of N, $A \neq 0$, there exists $x \in A$ such that $x \neq 0$. Hence there exists $r \in R$ such that $xr \neq 0$ and $xr \in M$. But $xr \in A$, so $M \cap A \neq 0$.

Thus $M \subseteq' N$.

Theorem 2.7: Given $A \subseteq B$, let C be maximal among submodules of B so that $A \cap C = 0$. Then $A \oplus C \subseteq' B$.

Proof: Assume C is maximal among submodules of B so that $A \cap C = 0$, and that $A + C$ is not large in B. Then there exists $D \subseteq B$ such that $(A + C) \cap D = 0$ with $D \neq 0$. Clearly $D \cap A = 0$ and $D \cap C = 0$, and $C \subsetneqq C + D$. Thus $(C + D) \cap A \neq 0$, so there exists $a \in A$ with $c + d = a$ for $c \in C$, $d \in D$. But $c + d = a$ implies $a - c = d$, and since $(A + C) \cap D = 0$ then $d = 0$, $a = c \in A \cap C = 0$ so $a = 0$. This is a contradiction. Therefore $A + C \subseteq' B$

Injectivity, essential extensions, and injective hulls.

Definition: Let N and M be R-modules. A module N extending M will be called an essential extension if M is large in N.

Theorem 2.8: Let N be an essential extension of M, and let I be an injective module containing M, then the injection map of M into I can be extended to a monomorphism of N into I.

Proof: Let $1: M \to I$ be the identity map from M into I, and let $\kappa: M \to N$ be a monomorphism. Then since I is injective, there exists $\phi: N \to I$ such that $I = \phi \circ \kappa$. We must show that $\phi: N \to I$ is a monomorphism. Now $\phi^{-1}(0) \cap M = 0$, and since M is large in N, $\phi^{-1}(0) \cap M = 0$ implies $\phi^{-1}(0) = 0$, or that ϕ is a monomorphism.

The above theorem states that if I is an injective module containing M, then I contains a copy of every essential extension of M.

Theorem 2.9: M is injective if and only if M has no proper essential extension.

Proof: $(\rightarrow)$ Assume M is injective, and let k be an essential extension of M. Since M is injective, M is a direct summand of K, and a direct summand can be essential only if is the whole module. Thus M = K, and M has no proper essential extension.

$(\leftarrow)$ Assume M has no proper essential extension, and let I be an injective module containing M. We wish to show that M is a direct summand of I, and hence injective. Let K be a submodule of I maximal with respect to the property that $K \cap M = 0$. We know that $K \subset M + K \subseteq' I$, and $m \cong (M + K)/K \subseteq I/K$. If we can show that $(M + K)/K \subseteq' I/K$, then since M has no proper essential extension, $(M + K)/K = I/K$, and thus $M + K = I$. Let $N/K \neq 0$. Then $N \supset K$, and thus $M \cap N \neq 0$, since k is maximal with the property that $K \cap M = 0$. Thus there exists $n \in N \cap M$, $n \neq 0$, and clearly $n + K \in (M + K)/K$ with $n + K \neq K$. Hence $(M + K)/K \subseteq' I/K$.

Definition: N is a maximal essential extension of M provided N is an essential extension of M and no proper extension of N is an essential extension of M.

We will show that every module has a maximal essential extension, and that this is unique in the following sense; given two maximal essential extensions N and N' of M, the identity mapping of M can be extended to an isomorphism from N onto N'.

We now define the injective hull of M.

Definition: The injective hull of M, denoted by $E(M)$, is an essential extension of M which is injective.

Several questions are immediately brought to mind by this definition. First, does an injective hull for every module exist and secondly, if it does exist, is it unique.

In order to see that an injective hull exists for every module, we consider the following method of construction. Every module M may be imbedded in an injective I. Thus we look at the collection of all essential extensions of M in I. These may be partially ordered by set inclusion. Since every chain has an upper bound, namely the union of all submodules in the chain, by Zorn's lemma, there exists a maximal one, call it K. We now show that K is injective. Since $M \subseteq' K \subseteq I$, by a previous theorem, any proper essential extension of K can be imbedded in I. Hence if K' is an essential extension of K, then $K \subseteq K' \subseteq I$. But since K' is an essential extension of M containing K, $K = K'$. Thus K has no proper essential extension, and is injective.

We now study the question of uniqueness. Assume $M \subseteq' I$ and $M \subseteq' I'$, with both I and I' injective. Then by a previous theorem $M \subseteq \bar{I} \subseteq' I'$, where $\bar{I} \cong I$. Since I is injective, $\bar{I}$ is a direct summand of I'. Therefore $I \cong \bar{I} = I'$, and the injective hull is unique up to isomorphism.

We follow this with a theorem which further characterizes injective hulls.

<u>Theorem 2.10</u>: Let N be an extension of M. Then the following statements are equivilent:

(1) N is a maximal essential extension of M.

(2) N is an essential extension of M and is injective.

(3) N is a minimal injective extension of M.

<u>Proof</u>: (1) $\rightarrow$ (2)

Let N be a maximal essential extension of M. Then N has no proper essential extension, and hence is injective by a previous theorem.

(2) $\rightarrow$ (3)

Let N be an essential extension of M which is injective. Then if I is any injective extension of M such that $M \subset I \subset N$, then N is an essential extension of I, and hence $N = I$. Thus N is a minimal injective extension of M.

$(3) \rightarrow (1)$

Let N be a minimal injective extension of M. Then since N is injective, N contains a copy of every essential extension of M. Thus $M \subseteq' N' \subset N$, where N' is a maximal essential extension of M in N. But by $(1) \rightarrow (2)$, N' is injective, so $N' = N$. Thus N is an essential extension of M. Since N is also injective it must be a maximal essential extension of M.

Theorem 2.11: Let $M \subseteq' N$. Then $E(M) = E(N)$.

Proof: By definition, $E(N)$ is an essential extension of N and is injective. Since $M \subseteq' N$ and $N \subseteq' E(N)$, then $M \subseteq' E(N)$. But $E(N)$ is injective, so $E(M) = E(N)$ by definition of injective hull.

CHAPTER III

INDECOMPOSABLE INJECTIVE MODULES

Definition: M is an indecomposable module provided M is not the direct sum of two non-zero submodules. If M is the direct sum of two non-zero submodules, then M is said to be decomposable.

Theorem 3.1: Let M be a right R-module, and let $\mathcal{A}$ be a non-empty collection of submodules of M. Then there is a set of submodules in $\mathcal{A}$ maximal with respect to the property that the sum of the set of modules is direct.

Proof: Let $\mathcal{A}$ be a set of submodules of M, and let $\mathcal{B}$ be the set of subsets of $\mathcal{A}$ with the property that the sum of any member of $\mathcal{B}$ is direct. Then $\mathcal{B}$ is non-empty, since singleton sets of submodules are in $\mathcal{B}$, and $\mathcal{B}$ can be partially ordered by set inclusion. Thus we let B be any chain in $\mathcal{B}$, and show that B has an upper bound in $\mathcal{B}$.

Let A be the union of the elements of B. Then if $A = \{M_i\}_{i\in I}$, A is clearly an upper bound on the elements of B. Thus we show that the sum of the $\{M_i\}_{i\in I}$ is direct, and hence $A \in \mathcal{B}$. Let $m \in \sum_{i\in I} M_i$, with $m = 0$. Then $m = \sum_{i\in I} m_i$, with all but a finite number of the $m_i = 0$. Let $m_{i_1}, m_{i_2}, \cdots, m_{i_k}$ be those elements of $\sum_{i\in I} m_i$ which are not equal to zero. Since B is a chain, with $m_{i_1} \in M_{i_1}$, $m_{i_2} \in M_{i_2}, \cdots, m_{i_k} \in M_{i_k}$, there exists an element A' of B with $\{M_{i_j}\}_{j=1}^{k} \subseteq A'$. Since the elements of A' form a direct sum, then

$\sum_{i=1}^{k} m_{i_k} = 0$ implies each of the $m_{i_k} = 0$. Thus $m = \sum_{i \in I} m_i = 0$ implies $m_i = 0$ for all $i \in I$, and hence the sum of the elements of A is direct. Thus $A \in \mathcal{B}$, and by Zorn's lemma, $\mathcal{B}$ has a maximal element.

Theorem 3.2: Let M be a right R-module. Then $E(M)$ is decomposable if and only if there exists $0 \neq A \subseteq M$ such that A is not large in M.

Proof: $(\rightarrow)$ Let $E(M) = A + B$, with $A, B \neq 0$, and $A \cap B = 0$. Since $M \subseteq' E(M)$, then $0 \neq M \cap A \subset M$, and $0 \neq M \cap B \subset M$. But $(M \cap A) \cap (M \cap B) = M \cap (A \cap B) = M \cap 0 = 0$. Thus M contains a non-zero submodule which is not large in M.

$(\leftarrow)$ Let $0 \neq A \subseteq M$ such that A is not large in M. Then $E(M)$ is not the injective hull of A, so there exists $E(A)$ such that $E(A) \not\subseteq E(M)$. By a previous theorem, since $E(M)$ is injective, $E(M)$ contains a copy of $E(A)$, call it $\overline{E(A)}$. But $\overline{E(A)}$ is injective so $E(M) = \overline{E(A)} + B$ where $B \neq 0$. Hence $E(M)$ is decomposable.

Definition: A module $M_R = 0$ is callen uniform if every non-zero submodule is large.

From the preceding theorem, we see that $E(M_R)$ is indecomposable if and only if M_R is uniform.

Theorem 3.3: M_R is an indecomposable injective, then $E = \mathrm{Hom}_R(M,M)$ is a local ring.

Proof: It is assumed that the reader knows that $E = \mathrm{Hom}_R(M,M)$ is a ring. Thus we must show that E is local. To do this, we use the fact that if $\lambda \in \mathrm{Hom}_R(M,M)$ is a monomorphism, then λ is a unit. Since only isomorphisms can be units, the question becomes one of showing that every monomorphism is an isomorphism. If $\lambda \in \mathrm{Hom}_R(M,M)$ is a monomorphism, then $\mathrm{Im}\lambda$ must be an injective copy of M. But M is indecomposable, so $\mathrm{Im}\lambda$ must be all of M. Thus λ is onto, and an isomorphism. We now return to the question of E being local. Let λ_1, λ_2 be non-units, and show $\lambda_1 + \lambda_2$ is a non-unit. Since λ_1, λ_2 are non-units, then λ_1, λ_2 are not monomorphisms, so $\ker \lambda_1 \neq 0$ and $\ker \lambda_2 \neq 0$. Since M_R is an indecomposable injective, then M_R is uniform. Hence $0 \neq \mathrm{Ker}\, \lambda_1 \cap \mathrm{Ker}\, \lambda_2 \subseteq \mathrm{Ker}\, (\lambda_1 + \lambda_2)$, and $\lambda_1 + \lambda_2$ is not a monomorphism. Therefore $\lambda_1 + \lambda_2$ is a non-unit.

There also exists a partial converse to the above theorem, which is stated below.

Theorem 3.4: Let M be an R-module with $\mathrm{Hom}_R(M,M)$ a local ring. Then M_R is indecomposable.

Proof: Assume false. Then M_R is decomposable, so $M_R = M_1 + M_2$, with $M_1 \cap M_2 = 0$. Let Π_1, κ_1 and Π_2, κ_2 be the canonical injection and projection maps respectively, for the direct sum. Define $\epsilon_1 = \kappa_1 \Pi_1 \,_1 \in \mathrm{Hom}_R(M,M)$, and $\epsilon_2 = \kappa_2 \Pi_2 \in \mathrm{Hom}_R(M,M)$. Clearly neither is a unit, since neither is onto. Also if $m_1 \in M_1$, $m_2 \in M_2$, then $\epsilon_1(m_1) = m_1$, and $\epsilon_1(m_2) = 0$ by definition. Thus we consider the sum $\epsilon_1 + \epsilon_2$. Let $m \in M$. Then $m = m_1 + m_2$, so

$$(\epsilon_1 + \epsilon_2)(m) = (\epsilon_1 + \epsilon_2)(m_1 + m_2) = (\epsilon_1 + \epsilon_2)(m_1) + (\epsilon_1 + \epsilon_2)(m_2)$$
$$= \epsilon_1(m_1) + \epsilon_2(m_1) + \epsilon_1(m_2) + \epsilon_2\,(m_2) = \epsilon_1(m_1) + \epsilon_2(m_2)$$
$$= m_1 + m_2 = m.$$

Thus $\epsilon_1 + \epsilon_2$ is the identity map, and hence a unit. Then $\mathrm{Hom}_R(M,M)$ is not local and this contradiction completes the proof.

Noetherian modules and injectivity

Definition: An R-module E is said to satisfy the ascending chain condition provided every ascending chain of submodules is finite.

Definition: A module which satisfies the ascending chain condition is said to be noetherian.

Theorem 3.5: If M_R is a noetherian module, then M_R contains a uniform submodule.

Proof: Let M be a noetherian module, and assume M contains no uniform submodule. Then there exists $A, B \in M$ such that $A \cap B = 0$ and $A, B \neq 0$. But B is not uniform, so there exists $B_{11}, B_{12} \subset B$ such that $B_{11} \cap B_{12} = 0$ and $B_{11}, B_{12} \neq 0$. Then for any finite number n, we can find $B_{n1}, B_{n2} \subset B_{n-1}, 2$ such that $B_{n1}, B_{n2} \neq 0$ and $B_{n1} \cap B_{n2} = 0$. We now form an ascending chain as follows:

$$A \subset A + B_{11} \subset A + B_{11} + B_{2}, \subset \cdots$$

Clearly this chain does not terminate, which contradicts the fact that M_R is noetherian. Thus M_R must contain a uniform submodule.

The following lemmas on noetherian modules are used to prove the next two theorems, and are stated without proof.

Lemma 3.6: If M_R is noetherian and $\alpha : M \to B$ is an epimorphism, then B is noetherian.

Lemma 3.7: If R is right noetherian and $M_R \neq 0$, then M_R has a uniform submodule.

Lemma 3.8: The direct product of a finite number of noetherian modules is noetherian.

<u>Definition</u>: A <u>ring R is noetherian</u> provided the module R_R is noetherian.

<u>Theorem 3.9</u>: If R is right noetherian and M_R is finitely generated, then M_R is noetherian.

<u>Proof</u>: Let M be a finitely generated module generated by the set $\{x_1, x_2, \cdots, x_n\}$ of n elements. Consider the direct product $R_R{}^n = R_R \times R_R \times \cdots \times R_R$ of n copies of R_R. By Lemma 3.8, $R_R{}^n$ is noetherian, and thus we need only define an epimorphism $\alpha : R_R{}^n \to M_R$ to show that M_R is noetherian. Let $\alpha : R_R{}^n \to M_R$ be defined by

$$\alpha(r) = \alpha(r_1, r_2, \cdots, r_n) = r_1x_1 + r_2x_2 + \cdots + {}_nx_n$$

where $r \in R_R{}^n$.

Then α is easily seen to be a homomorphism, and is onto by definition. Thus by Lemma 3.6, M_R is noetherian.

<u>Theorem 3.10</u>: If R is right noetherain and $\{M_i \mid i \in I\}$ is a collection of injective modules, then $\coprod_{i \in I} M_i$ is injective.

<u>Proof</u>: Assume that R is right noetherain and let $M = \coprod_{i \in I} M_i$. We wish to show that M is injective. Let K be a right ideal of R. Then $K \subseteq R$, and M_i is injective, so for any $\phi : K \to M$, we can find a homomorphism $\beta_i : R \to M_i$ with $\beta_i \circ \alpha = \Pi_i \circ \phi$ for each $i \in I$.

$$\begin{tikzcd}
M_i \arrow[dr, shift left, "\epsilon_i"] & & R \arrow[ll, "\beta_i"'] \\
& M \arrow[ul, shift left, "\Pi_i"] & \\
& & K \arrow[uu, "\alpha"'] \arrow[ul, "\phi"]
\end{tikzcd}$$

By Baer's lemma, there exists $m_i \in M_i$ such that for any $k \in K$, $\Pi_i \circ \phi(k) = m_i k$. Since R is right noetherian and $K \subset R$, then K is finitely generated. Thus there exists a finite set $F \subset I$ so that $\Pi_i \circ \phi(k) \neq 0$ only if $i \in F$. Hence

$$M \cong \coprod_{i \in F} M_i \times \coprod_{i \notin F} M_i \cong \sum_{i \in F} \kappa_i(M_i) \oplus \sum_{i \notin F} \kappa_i(M_i).$$

Since $\coprod_{i \in F} M_i$ is a direct sum of a finite number of modules, then $\coprod_{i \in F} M_i = \prod_{i \in F} M_i$, and $\prod_{i \in F} M_i$ is injective by Theorem 2.1. Hence by Baer's lemma, for $\phi: K \to \prod_{i \in F} M_i \cong \sum_{i \in F} \kappa_i(M_i)$, there exists $m \in \prod M_i$ such that $\phi(k) = mk$. But since ϕ was arbitrary and $\prod M_i \subset M$, then M is injective.

Theorem 3.11: If R is right noetherian and I_R is injective, then I_R is a direct sum of indecomposable injectives.

Proof: Let S be the set of all indecomposable injective submodules of I_R. Then S is non-empty, since by Lemma 3.7, I_R contains a uniform submodule, and hence an indecomposable injective module. By Theorem 3.1, there is an $S' \subset S$ such that S' is maximal with respect to the property that the sum of the submodules is direct. Let A be equal to this maximal direct sum. Then A is injective, so $I = A \oplus B$. If $B = 0$, then we are through. Thus assume $B \neq 0$. then B is an R-module, so B contains a uniform submodule, and hence an indecomposable injective submodule B'. But $A \cap B' = 0$, so the sum of the elements of $S' \cup \{B'\}$ is direct, with $S' \subset S' \cup \{B'\}$. This contradicts the fact that S' is maximal. Thus $B = 0$ and I_R is a direct sum of indecomposable injectives.

The above theorem shows that any injective R-module, where R is a commutative noetherain ring, may be written as a direct sum of indecomposable injective modules. The uniqueness of this direct sum follows from the Azumaya - Krull - Schmidt theorem.

CHAPTER IV

INDECOMPOSABLE INJECTIVES OVER COMMUTATIVE NOETHERIAN RINGS

From our previous theorems, we see that $E(M)$ is an indecomposable injective precisely when M is uniform. Thus $E(M)$ is also the injective hull for any non-zero submodule of M. Since every non-zero module contains a non-zero cyclic submodule, then every indecomposable injective is the injective hull of a cyclic module. Any cyclic submodule of M is of the form xR for some $x \in M$. Since $x:R \to xR$ defined by $r \to xr$ is easily seen to be an epimorphism with $\ker \hat{x} = \{xR \mid xr = 0\} \equiv \text{ann } x$, where ann x is called the annihilator ideal of x, then from the first isomorphism theorem, $xR = R/\text{ann } x$. Thus any cyclic submodule of M generated by x is isomorphic to R/I, where $I = \text{ann } x$. This brings us to the following conclusion:

<u>All indecomposable injective look like $E(R/I)$</u>.

<u>Theorem 4.1</u>: Let R be a commutative noetherian ring and let I be an ideal of R. Then an R-module M contains a copy of R/I if and only if there exists an $x \in M$ such that $\text{ann } x = I$.

<u>Proof</u>: ($\to$) Assume M contains a copy of R/I. Then there exists a monomorphism $\alpha:R/I \to M$. Since R/I is cyclic and generated by $(1 + I)$, then there exists an $x \in M$ such that $\alpha(1 + I) = x$. Now for any $r \in R$, $\alpha(1 + I)r = 0$ precisely when $r \in I$. Thus $\alpha(1 + I)r = xr = 0$ for exactly those $r \in I$ implies $I = \text{ann } x$.

($\leftarrow$) Assume there exists an $x \in M$ such that $\text{ann } x = I$. Then we can define a mapping $\hat{x}:R \to M$ by $r \to xr$. Clearly $\hat{x}$ is a homomorphism, and $\text{Ker } \hat{x} = \text{ann } x = I$. Thus $\hat{x}:R \to \text{Im } \hat{x}$ is an epimorphism, so $R/I \cong \text{Im } \hat{x}$ where $\text{Im } \hat{x} \subseteq M$. Thus M contains a copy of R/I.

We now consider the following question: What kind of ideals are necessary for R/I to be uniform? Clearly R/I is not uniform if and only if for two ideals K, L of R, I is properly contained in K and L, and $K \cap L = I$. Thus R/I is uniform if and only if I is intersection-irreducible; that is, I is not the intersection of two ideals which properly contain I. All prime ideals are intersection-irreducible. We formalize these statements in the theorems below.

<u>Theorem 4.2</u>: R/I is uniform if and only if I is intersection-irreducible.

<u>Proof</u>: ($\rightarrow$) Let R/I be uniform. Then if K, L are two ideals of R containing I, then $K/I \cap L/I \neq 0$. Thus there exists $x \in K, L$ such that $x \notin I$. Therefore $K \cap L \neq I$, and I is intersection-irreducible.

($\leftarrow$) Let I be intersection-irreducible. Then if K, L properly contain I, $K \cap L \neq I$. Thus $K/I \cap L/I \neq 0$, and R/I is uniform.

<u>Theorem 4.3</u>: Any prime ideal is intersection-irreducible.

<u>Proof</u>: Let P be a prime ideal, with K, L ideals of a ring R. Then if $K \cap L = P$, since $KL \subseteq K \cap L = P$, either $K \subseteq P$ or $L \subseteq P$. Thus P is intersection-irreducible.

For the following theorems, R is assumed to be a commutative noetherian ring, P a prime ideal of R, and E an indecomposalbe injective module.

Theorem 4.4: If M is a non-zero R-module and P is maximal among ideals $\{\text{ann } x \mid x \in M, x \neq 0\}$, then P is prime.

Proof: Let $ab \in P$, with $a \notin P$. Then we wish to show that $b \in P$. Since P is maximal among $\{\text{ann } x \mid x \in M, x \neq 0\}$, then $P = \text{ann } x$ for some $x \in M$. Thus if $ab \in P$, then $(ab)x = 0$. Now $(ab)x = 0$ implies $b(ax) = 0$ since R is commutative. But $a \notin P$, so $ax \neq 0$. Thus $b \in P_1$, where $P_1 = \text{ann } ax$. If $c \in P$, then $cx = 0$, so $c(ax) = a(cx) = 0$. Thus $P \subset P_1$. But P is maximal, so $P = P_1$. Thus $b \in P$, and P is prime.

Definition: Let I be an ideal of a commutative ring R. Then the radical of I, denoted $\sqrt{I}$, is the set of all $x \in R$ such that $x^n \in I$ for some $n \geq 1$. We note that $I \subseteq \sqrt{I}$.

Theorem 4.5: Let E be an indecomposable injective module.

(1) There is a prime ideal P so that $E \cong E(R/P)$.

(2) If P_1 and P_2 are prime ideals and $E(R/P_1) \cong E(R/P_2)$, then $P_1 = P_2$.

(3) If $E = E(R/P)$ and $x \in E - 0$, then $\sqrt{\text{ann } x} = P$.

Proof of (1): Consider the set $\mathcal{A} = \{\text{ann } x \mid x \in E, x \neq 0\}$. Then if there exists an ideal P such that P is maximal in $\mathcal{A}$, then by Theorem 4.4, P is prime. Since $P = \text{ann } x$ for some $x \in E$, by Theorem 4.1, E contains a copy of R/P, and thus $E(R/P) \cong E$. Thus we need only show that P exists.

Clearly $\mathcal{A}$ is partially ordered by set inclusion. Thus consider an ascending chain of elements of $\mathcal{A}$, say

$$\text{ann } x_1 \subseteq \text{ann } x_2 \subseteq \text{ann } x_3 \subseteq \cdots .$$

Since R is noetherian, there exists an n such that $\operatorname{ann} x_n = \operatorname{ann} x_{n+1}$, or $\operatorname{ann} x_n$ is an upper bound. But $\operatorname{ann} x_n \in A$, so by Zorn's Lemma, A has a maximal element. Thus P exists.

Proof of (2): Assume $E \cong E(R/P_1) \cong E(R/P_2)$. Then by Theorem 4.1, there exists $x_1, x_2 \in E$ such that $\operatorname{ann} x_1 = P_1$ and $\operatorname{ann} x_2 = P_2$. Thus $x_1R \cong R/P_1$ and $x_2R \cong R/P_2$. Since E is uniform, $x_1R \cap x_2R \neq 0$, so there exists $x_1r_1 = x_2r_2 \neq 0$. For any $p \in P_1$, $p_1x_1r_1 = 0$, so $p_1x_2r_2 = 0$. But this says that $P_1 \subseteq \operatorname{ann} x_1r_1 \subseteq P_2$. Similarly, $P_2 \subseteq \operatorname{ann} x_2r_2 \subseteq P_1$. Thus $P_1 = P_2$, and the prime ideal P in (1) is unique.

Proof of (3): We first show that $\operatorname{ann} x \subseteq P$. Let $x \in E$, and consider the collection of annihilator ideals which contain $\operatorname{ann} x$. Then this collection has a maximal element, say $\operatorname{ann} x'$. Since $\operatorname{ann} x'$ is maximal among annihilators, $\operatorname{ann} x'$ is prime, and hence $E = E(R/\operatorname{ann} x')$. But $E = E(R/P)$, so by (2), $P = \operatorname{ann} x'$. Thus $\operatorname{ann} x \subseteq P$.

We now show that $P \subseteq \sqrt{\operatorname{ann} x}$. Assume there exists $r \in P$ such that $r^k \notin \operatorname{ann} x$ for all $k \geq 0$. Then define $T_k = \{s \in R \mid r^k s \in \operatorname{ann} x\} = \{s \in R \mid xr^k s = 0\}$. Then $T_1 \subseteq T_2 \subseteq T_3 \subseteq \ldots$ is an ascending chain of ideals, and since R is noetherian, there exists an n such that $T_n = T_{n+1}$. Since $xr^n \neq 0$ and E is uniform, $xr^n R \cap R/P \neq 0$, or there exists $s \in R$ such that $0 \neq xr^n s \in R/P$. But since any element of R/P is annihilated by all P elements, and $r \in P$, then $(xr^n s) r = 0$ implies $xr^{n+1} s = 0$. Thus $s \in T_{n+1} - T_n$ implies $T_{n+1} \neq T_n$. This is a contradiction, so $P \subseteq \sqrt{\operatorname{ann} x}$. Thus we have $P = \sqrt{\operatorname{ann} x}$.

Remark: Part (3) shows that P is the unique largest ideal among $\{\text{ann } x \mid x \in E, x \neq 0\}$.

The importance of the above theorem cannot be overstated. It shows that there is a one-to-one correspondence between prime ideals and isomorphism classes of indecomposable injectives for a commutative noetherian ring R. Let us examine this to see how it works. Given any indecomposable injective R-module, we can find a prime ideal P by taking the unique largest ideal of the set $\{\text{ann } x \mid x \in E, x \neq 0\}$. This ideal exists and is unique by Theorem 4.5. Thus there exists a one-to-one mapping from the set of indecomposable injective R-modules to the set of isomorphism classes of prime ideals of R. Consider now any prime ideal P of R. Then R/P is uniform by Theorems 4.2 and 4.3, so $E(R/P)$ is indecomposable. Again by Theorem 4.5, P is unique, so we have a one-to-one mapping from the set of isomorphism classes of prime ideals of R to the set of indecomposable injective R-modules. Thus there is a one-to-one correspondence between the two sets. The importance of this is obvious. We have now characterized up to isomorphism all indecomposable injective R-modules as the injective hull of the R-module R/P, where P is a prime ideal.

Theorem 4.6: Let $E = E(R/P)$, with P a prime ideal of R, and let $A_k = \{x \in E \mid xP^k = 0\}$. Then $E = \bigcup_{k=0}^{\infty} A_k$.

Proof: Clearly $\bigcup_{k=0}^{\infty} {}_k \subseteq E$. Thus we show that $E \subseteq \bigcup_{k=0}^{\infty} A_k$. Let $x \in E$, and consider $\text{ann } x$. If we can show that there exists a $k \in N$ such that $P^k \subseteq \text{ann } x$, then $x \in A_k$, so $x \in \bigcup_{k=0}^{\infty} A_k$. From Theorem 4.5 (3), we have $\sqrt{\text{ann } x} = P$. Thus if $p \in P$, there exists a k such that $p^k \in \text{ann } x$. Since R is noetherian, P is finitely generated, say by the set $\{a_1, a_2, \ldots, a_n\}$. For each a_i, there exists k_i such that $a_i^{k_i} \in P$. Let $\ell = \mathrm{lcm}(k_1, k_2, \ldots, k_n)$. Then for $P^{n\ell}$, we have $P^{n\ell} \subseteq \mathrm{ann}\ x$. Thus $\bigcup_{k=0}^{\infty} A_k = E$.

<u>The indecomposable injective modules of Z.</u>

Let us consider the ring of integers under the usual addition and multiplication. Then $(Z, +, \cdot)$ is a commutative noetherian ring, so we proceed to use our previous theorems in an attempt to characterize all indecomposable injective Z-modules. We recall that there is a one-to-one correspondence between prime ideals and indecomposable injectives, so the set $\{E(Z/P) \mid P \text{ is a prime ideal of } Z\}$ is the set of all indecomposable injectives over Z. Since all prime ideals of Z are either 0 or pZ for some prime p, then each indecomposable injective will be either $E(Z)$ or $E(Z/pZ)$.

Consider Q as a Z-module. Since Q is divisible, then by Theorem 2.3, Q_Z is injective. Clearly Q is an extention of Z, so we ask if Q is an essential extension. In order to demonstrate that it is, we show (1) Z is large in Q, and (2) Z is uniform. Let Q' be a non-zero submodule of Q. Then Q' has a non-zero element p/q. Since Q is closed under Z-multiplication, $qp/q = p$ is a non-zero element of Q', which says that $Q' \cap Z \neq 0$. Since Q' was arbitrary, Z is large in Q. To show Z is uniform, let Z', Z'' be non-zero submodules of Z. Since Z is cyclic, Z', Z'' are cyclic, so $Z' = pZ$, $Z'' = qZ$ for some $p, q \neq 0$. Thus $pq \in Z'$, $qp \in Z''$ implies $Z' \cap Z'' \neq 0$. Thus Z is uniform, and $Q_Z = E(Z)$.

We now consider the injective hull of Z/pZ. Define $Z_{p^\infty} = (Q/Z)p = \{\frac{a}{p^k} + Z \mid a \in Z,\ k \geq 0\}$. We shall denote the coset $\frac{a}{p^k} + Z$ by $\overline{\frac{a}{p^k}}$, with $\frac{a}{p^k}$ reduced to lowest terms. We would like to show that $E(Z/pZ) = Zp^\infty$. It is apparent that $\alpha: Z/pZ \to Zp^\infty$ defined by $\alpha(a) = (\overline{\frac{a}{p}})$ is a monomorphism, and thus Zp^∞ is an extension of Z/pZ. Thus we would like to show it is an essential extension and injective. Let H be any non-zero submodule of Zp^∞. Then if $\overline{\frac{a}{p^k}} \in H$, since H is closed under Z-multiplication, $p^{k-1}(\overline{\frac{a}{p^k}}) = \overline{\frac{a}{p^1}} \in H$. Since $0 \neq \bar{a} \in Z/pZ$, then $H \cap Z/pZ \neq 0$, and Zp^∞ is an essential extension of Z/pZ. To show that Zp^∞ is injective, we will show it is divisible. Let $\overline{\frac{a}{p^k}} \in Zp^\infty$, with $z \in Z$. We need to show that there exists $x' \in Zp^\infty$ such that $zx' = \overline{\frac{a}{p^k}}$. Consider the following two cases: (1) $(z, p^k) = 1$ and (2) $(z, p^k) = p^\ell$ for some $\ell \leq k$. For case (1), if $(z, p^k) = 1$, then there exists $x, y \in Z$ such that $xz + yp^k = 1$. Thus $(xa)z + (ya)p^k = a$, so $\overline{\frac{z(xa) + (ya)p^k}{p^k}} = \overline{\frac{a}{p^k}}$.
But this implies

$$\overline{\frac{z(xa) + (ya)p^k}{p^k}} = \overline{\frac{z(xa)}{p^k}} + \overline{\frac{(ya)p^k}{p^k}} = \overline{\frac{z(xa)}{p^k}} = \overline{\frac{a}{p^k}}.$$ Thus $x' = \overline{\frac{xa}{p^k}}$.

For case (2), if $(z, p^k) = p^\ell$ for some $\ell \leq k$, then $z = z'p^\ell$ where $(z', p^k) = 1$. As with case (1), we have that $\overline{\frac{z'(xa)}{p^k}} = \overline{\frac{a}{p^k}}$, and hence

$$\overline{\frac{p\ (z')(xa)}{p^{k+\ell}}} = \overline{\frac{z(xa)}{p^{k+\ell}}} = \overline{\frac{a}{p^k}},$$ and thus $x' = \overline{\frac{xa}{p^{k+\ell}}}$. Thus

Zp^∞ is divisible.

We shall examine the structure of Zp^∞. From Theorem 4.6, we know that $Zp^\infty = \bigcup_{k=0}^{\infty} A_k$, where $A_k = \{x \in Zp^\infty \mid x(pZ)^k = 0\}$.

Thus

$$A_0 = \frac{\bar{1}}{p^0} = \{\frac{a}{p^0} \mid a \in Z\} = \{0\}$$

$$A_1 = \frac{\bar{1}}{p^1} = \{\frac{a}{p^1} \mid a \in Z\}$$

$$\vdots$$

$$A_k = \frac{\bar{1}}{p^k} = \{\frac{a}{p^k} \mid a \in Z\}$$

Clearly $A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_k \subset \ldots$ with $\bigcup_{k=0}^{\infty} A_k = Zp^\infty$.

We now show that Zp^∞ has no other subgroups than these. Let H be a proper subgroup of Zp^∞ such that $H \neq A_k$ for any $k > 0$. Then $A_0 \subset H$, and there exists a smallest positive integer $n \geq 1$ such that $A_n \not\subseteq H$. Thus $A_{n-1} \subset H$, and we wish to show that $A_{n-1} = H$. Assume there exists an $\frac{\bar{a}}{p^\ell} \in H - A_{n-1}$, where $\ell \geq n$. Then since $(a, p^n) = 1$, $(a, p) = 1$, so there exists an $x, y \in Z$ such that $ax + py = 1$. Thus we have $\frac{\bar{1}}{p^\ell} = \frac{\overline{ax}}{p^\ell} + \frac{\overline{py}}{p^\ell}$. If $n = \ell$, then

$\frac{\overline{ax}}{p^\ell} = \frac{\overline{ax}}{p^n} \in H$, and $\frac{\overline{py}}{p^\ell} = \frac{\overline{py}}{p^n} = \frac{\bar{y}}{p^{n-1}} \in H$, so $\frac{\bar{1}}{p^\ell} \in H$. But this implies that $A_n \subset H$, since A_n is generated by $\frac{\bar{1}}{p^n}$, which is a contradiction. If $k > n$, then $\ell - n = j > 0$, so multiplying by p^j, we have

$p^j \left(\frac{\bar{1}}{p^\ell}\right) = p^j \left(\frac{\overline{ax}}{p^\ell} + \frac{\overline{py}}{p^\ell}\right)$ implies that $\frac{1}{p^n} = \frac{\overline{ax}}{p^n} + \frac{\overline{py}}{p^n}$. Since

$\frac{\overline{ax}}{p} \in H$, then $p^j \left(\frac{\overline{ax}}{p^\ell}\right) = \frac{\overline{ax}}{p^n} \in H$, and for

$\frac{\overline{py}}{p^\ell}$, $p^j\left(\frac{\overline{py}}{p^\ell}\right) = c\left(\frac{\overline{py}}{p^n}\right) = \frac{\bar{y}}{p^{n-1}} \in H$, so $\frac{\bar{1}}{p^n} \in H$, which is again a

contradiction. Thus $\{A_k\}_{k=0}^{\infty}$ are the only proper subgroups of Zp^∞.

We have just characterized all indecomposable injective modules of Z; i.e., either they are Q_Z or Z_{p^∞} for some prime p. But our result is even stronger than this. Using Theorem 3.11, we have characterized all injective Z-modules, since every injective Z-module is the direct sum of indecomposable injectives.

BIBLIOGRAPHY

1. Joachim Lambek, Lectures on Rings and Modules, Blaisdell Publishing Company, Waltham, Massachusetts, 1966.

2. Irving Kaplansky, Infinite Abelian Groups, The University of Michigan Press, Ann Arbor, Michigan, 1969.

3. Douglas Geoffrey Northcott, Lessons on Rings, Modules and Multiplicities, Cambridge University Press, London, 1968.

4. D. W. Sharpe and P. Vãmos, Injective Modules, Cambridge University Press, Cambridge, 1972.